

# Emergent Threebrane Lattices

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**ABSTRACT:** In this article the anomalous dimension of a class of operators with a bare dimension of  $O(N)$  is studied. The operators considered are dual to excited states of a two giant graviton system. In the Yang Mills theory they are described by restricted Schur polynomials, labeled with Young diagrams that have at most two columns. In a certain limit the dilatation operator looks like a lattice version of a second derivative, with the lattice emerging from the Young diagram itself.

**KEYWORDS:** Giant Gravitons, AdS/CFT correspondence, super Yang-Mills theory.

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## 1. Introduction

The AdS/CFT correspondence[1] is a concrete realization of the conjectured equivalence[2] between gauge theory and gravity, in 't Hooft's large  $N$  limit. The identification of bulk strings with certain operators in the boundary theory was accomplished in [3]. The relevant operators take the form

$$\text{Tr}(YZZZZYZZZZZZYZZZZZY)$$

A useful way of thinking about these operators, is that the  $Z$ s form a lattice on which the impurities ( $Y$ ) hop. Semiclassical states can be obtained by putting each lattice site in a

coherent state. It is possible to derive a sigma model action which describes the relevant semiclassical dynamics. This action matches the low energy limit of the Polyakov action, providing a striking match between the string and Yang-Mills dynamics[4, 5, 6]. What are the ingredients necessary for this matching?

- As the  $\mathcal{R}$ -charge  $J$  of the operators increase, they describe states in the string theory of an increasing angular momentum. These states will expand as a consequence of the Myers effect. The operators considered should contain  $J = O(\sqrt{N})$   $Z$ -fields and  $O(1)$  impurities if they are to expand to string sized objects.
- The dilatation operator closes on a set of single trace operators. To prevent mixing with multitraces, it is necessary to take  $\frac{J^2}{N} \ll 1$ . Notice that this is consistent with taking  $J = O(\sqrt{N})$ .
- The dilatation operator acts (for example) on the second  $Y$  in

$$\mathcal{O}_l = \text{Tr}(Y Z^l Y Z^{J-l})$$

to produce the combination  $\propto \mathcal{O}_{l+1} + \mathcal{O}_{l-1} - 2\mathcal{O}_l$ . There are two things worth noting. First, mixing between the operators  $\mathcal{O}_l$  is highly constrained. At large  $N$  we have  $\langle \mathcal{O}_l \mathcal{O}_k^\dagger \rangle \propto \delta_{kl}$ . To one loop,  $\mathcal{O}_l$  can only mix with  $\mathcal{O}_{l\pm 1}$ . Second, the above linear combination clearly provides a lattice approximation to a second derivative. In this way the interpretation of the  $Z$ s as providing a lattice is natural and we see concretely how the string worldsheet emerges from the Yang-Mills theory.

This demonstration of stringy degrees of freedom in the Hilbert space of super Yang-Mills theory is encouraging, but is not a complete story. Indeed, since the discovery of D-branes[7] it has been clear that there is more to string theory than strings - there are membrane excitations of various dimensionality in the theory. If the operator in the Yang-Mills theory is to describe a threebrane of size  $\approx 1$  in units of the radius of curvature of the AdS space, we must consider operators with an  $\mathcal{R}$ -charge of  $O(N)$ . These are the so-called giant gravitons[8]. For operators with such a large  $\mathcal{R}$ -charge there will be uncontrolled mixing between different trace structures, as a consequence of exploding combinatoric factors that over power the usual  $\frac{1}{N}$  suppression of non-planar terms[9]. This difficulty was solved in [10] where it was shown that the Schur polynomials have diagonal two point functions to all orders in  $\frac{1}{N}$ . The Schur polynomials are labeled by Young diagrams. The picture that naturally emerges[10] (see also [11]) is that a Young diagram with  $n$  long columns (a long column has  $O(N)$  boxes in it) is dual to a state of  $n$  giant gravitons that have expanded in the  $S^5$  of  $\text{AdS}_5 \times S^5$ ; a Young

diagram with  $n$  long rows is dual to a state of  $n$  giant gravitons that have expanded in the  $\text{AdS}_5$  of  $\text{AdS}_5 \times S^5$ .

Given that we know the operators dual to giant gravitons, it seems natural to ask if we can compute the anomalous dimension of these operators. Further, by repeating the argument that worked for strings, does the geometry of the threebrane<sup>1</sup> world volume emerge? Concretely, by acting with the dilatation operator on an operator dual to a giant graviton, does one see any hint of the giant worldvolume geometry? There are a number of problems that need to be solved before this can be carried out:

- One needs to include more than one matrix; all operators that are built out of a single complex Higgs field preserve one half of the supersymmetries and hence are annihilated by the dilatation operator. The Schur polynomials are built out of  $Z$ s only. Fortunately, there is a multi-matrix generalization of the Schur polynomials, the restricted Schur polynomials[12, 13, 14, 15, 16]. The restricted Schur polynomial built out of  $p$  matrices is labeled by  $p + 1$  Young diagrams. Enhanced global non-abelian symmetries at zero coupling in Yang Mills theory provide a useful understanding (which generalizes to other bases - see below) of how the restricted Schur polynomials diagonalize the two-point functions of these multi-matrix operators[17].
- One would need to study more than just a single threebrane. Indeed, the small fluctuations of a single threebrane[18, 19] do not break supersymmetry, so consequently we expect any Schur polynomial whose Young diagram labels have a single column or a single row (corresponding to a one threebrane state) will be annihilated by the dilatation operator. We will explicitly demonstrate this.
- The results from[18] also suggest another difficulty: you don't see radius dependence in the spectrum. This is not at all what you would expect. Indeed, as the threebrane grows, you'd expect the wavelength of the vibration modes to increase and consequently the energy of the mode to decrease. This naive expectation is not quite correct because the threebrane expands in a nontrivial geometry. Due to the geometry, the modes of the larger threebrane are blue shifted. This blue shifting exactly compensates the larger wavelength so that the spectrum becomes independent of the size of the giant graviton.
- Taking the large  $N$  limit usually provides a significant simplification, because non-planar diagrams can be neglected. For the problem at hand it is not obvious if there is any simplification.

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<sup>1</sup>In this article the only threebrane we consider is a giant graviton. Thus all threebranes have  $S^3$  topology.

We would like to study the simplest possible case of an excited giant graviton state. It is clear that many features, including the emergence of the worldsheet lattice could be seen by considering the case of a BMN operator with just two impurities. With this motivation, in this article we study operators with an  $\mathcal{R}$ -charge of order  $N$  that contain two impurities. Concretely, we built the operators using  $n$   $Z$ s and 2  $Y$ s, with  $n = O(N)$ . The operators are restricted Schur polynomials labeled by a Young diagram with  $n+2$  boxes, a Young diagram with  $n$  boxes and a Young diagram with 2 boxes. Both the Young diagram with  $n+2$  boxes and the one with  $n$  boxes have two large columns, so that we are studying a two threebrane state. The threebrane state carries two angular momenta. In the  $Z$  plane it carries  $O(N)$  units of angular momentum whilst in the  $Y$  plane it carries two units. When visualizing the two threebrane state, the angular momentum in the  $Y$  plane can be neglected so that the fluctuating threebranes will remain spherical, i.e. only the radius of the giant graviton can undergo fluctuations. The giants will interact by means of open strings stretched between them. For this simple system, it is possible to anticipate the results. Each threebrane will behave like a harmonic oscillator; the open strings stretched between these threebranes implies that the oscillators are coupled. It is well known that coupled oscillators have two possible normal modes, corresponding to the oscillators oscillating in phase or out of phase. We will see that this physical picture does indeed emerge.

There are a number of papers discussing topics that are related to our study. A related paper [20] with very similar goals, considers near maximal giant gravitons and their open string fluctuations at large  $N$ . For studies of strings attached to giant gravitons see [12, 22, 13, 14, 15]. The basis provided by the restricted Schur polynomials is only one of a number of possible bases. For other bases see [23, 24, 25, 17, 26]. The one loop dilatation operator in these bases has been studied in [25, 27]. For a useful recent review of this material see [28]. Finally, the papers [29] apply similar techniques to study operators of dimension  $O(\sqrt{N})$ . These operators are dual to strings.

## 2. Action of the Dilatation Operator

We will consider the action of the one loop dilatation operator in the  $SU(2)$  sector[30]

$$D = g_{\text{YM}}^2 \text{Tr} [Y, Z][\partial_Y, \partial_Z]$$

on the restricted Schur polynomial

$$\chi_{(R,(r,s))}(Z^{\otimes n}, Y^{\otimes 2}) = \frac{1}{n!2!} \sum_{\sigma \in S_{n+2}} \text{Tr}_{(r,s)}(\Gamma_R(\sigma)) Z_{i_{\sigma(1)}}^{i_1} \cdots Z_{i_{\sigma(n)}}^{i_n} Y_{i_{\sigma(n+1)}}^{i_{n+1}} Y_{i_{\sigma(n+2)}}^{i_{n+2}}.$$

The labels of our restricted Schur polynomial  $\chi_{(R,(r,s))}$  are (i)  $R$ , which is a Young diagram with  $n+2$  boxes or equivalently an irreducible representation of  $S_{n+2}$ , (ii)  $r$ , which is a Young diagram with  $n$  boxes or equivalently an irreducible representation of  $S_n$  and (iii)  $s$  which is a Young diagram with 2 boxes or equivalently an irreducible representation of  $S_2$ . A simple calculation yields

$$D \chi_{(R,(r,s))}(Z^{\otimes n}, Y^{\otimes 2}) = \frac{g_{\text{YM}}^2}{(n-1)!} \sum_{\psi \in S_{n+2}} \text{Tr}_{(r,s)} (\Gamma_R((n, n+2)\psi - \psi(n, n+2))) \times \\ \times Z_{i_{\psi(1)}}^{i_1} \cdots Z_{i_{\psi(n-1)}}^{i_{n-1}} Y_{i_{\psi(n+1)}}^{i_{n+1}} (YZ - ZY)_{i_{\psi(n)}}^{i_n} \delta_{i_{\psi(n+2)}}^{i_{n+2}}. \quad (2.1)$$

This can also be expressed as

$$D \chi_{(R,(r,s))}(Z^{\otimes n}, Y^{\otimes 2}) = g_{\text{YM}}^2 \text{Tr} \left( \frac{d}{dV} \right) \frac{1}{(n-1)!} \sum_{\psi \in S_{n+2}} \text{Tr}_{(r,s)} (\Gamma_R([(n, n+2), \psi])) \times \\ \times Z_{i_{\psi(1)}}^{i_1} \cdots Z_{i_{\psi(n-1)}}^{i_{n-1}} U_{i_{\psi(n)}}^{i_n} Y_{i_{\psi(n+1)}}^{i_{n+1}} V_{i_{\psi(n+2)}}^{i_{n+2}},$$

where

$$U = YZ - ZY.$$

We would like to express this result as a sum over restricted Schur polynomials. The basis that we are using is obtained by choosing an  $S_n \times S_2$  subgroup of  $S_{n+2}$ . One way to construct the relevant representation of the subgroup, is to remove two boxes from  $R$  to obtain  $r$ ; the two boxes which are removed can then be arranged into irreducible representations of  $S_2$ . There is a second basis, which employs an  $S_n \times S_1 \times S_1$  basis. In this second basis, one simply keeps track of the order in which boxes were removed. The removed boxes are not arranged into irreducible representations of  $S_2$ .

In the  $S_n \times S_1 \times S_1$  basis it is straight forward to evaluate the action of  $(n, n+2)$  in the  $\text{Tr}_{(r,s)} (\Gamma_R([(n, n+2), \psi]))$  factor [15]. The action of the derivative in this basis has been worked out in [31, 13]. Finally, we need to “separate” the products of  $X$  and  $Z$  appearing in  $U$ . This can be done using the methods developed in [14, 15]. We will carry out this computation exactly, that is, to all orders in  $\frac{1}{N}$ .

The computation we performed can be summarized as

- Change from the  $S_n \times S_2$  basis to the  $S_n \times S_1 \times S_1$  basis. The formulas for this change of basis are given in an Appendix.
- Evaluate the action of  $(n, n+2)$ . The resulting character identities are given in an Appendix.

- It is clear that the  $S_n \times S_1 \times S_1$  basis is more convenient for actually performing the computation. Why not simply stick to this basis from the start? The  $S_n \times S_2$  basis must be used, because it is only in this basis that we obtain a basis of operators whose free two point functions are orthogonal to all orders in  $\frac{1}{N}$ . In fact, the  $S_n \times S_1 \times S_1$  basis is over complete; it would be the correct basis if the two impurities were different fields.

Single threebrane states are given by choosing  $R$  to be a Young diagram with a single column (for a giant graviton that has expanded in the  $S^5$  of  $\text{AdS}_5 \times S^5$ ) or a single row (for a giant graviton that has expanded in the  $\text{AdS}_5$ ). These are both one dimensional representations of the symmetric group. Hence, the restrictions are trivial and the representation is obviously Abelian. This implies that

$\text{Tr}_{(r,s)}(\Gamma_R([(n, n+2), \psi])) = \text{Tr}_{(r,s)}(\Gamma_R((n, n+2))\Gamma_R(\psi) - \Gamma_R(\psi)\Gamma_R((n, n+2))) = 0$   
so that (2.1) vanishes. Thus, single threebrane states are supersymmetric. This is in perfect agreement with the worldvolume analysis of [18, 19].

The general two threebrane states are given by the following operators

<sup>2</sup>Recall that the weight of a box in row  $i$  and column  $j$  is  $N - i + j$ .

$$O_d(b_0, b_1) = \chi_{\begin{array}{|c|} \hline \square \\ \square \\ \square \\ \vdots \\ \square \\ \square \\ \square \\ \hline \end{array}}, \quad O_e(b_0, b_1) = \chi_{\begin{array}{|cc|} \hline \square & \square \\ \square & \square \\ \square & \square \\ \vdots & \vdots \\ \square & \square \\ \square & \square \\ \square & \square \\ \hline \end{array}}.$$

Once the representation of  $S_2$  and the columns in  $R$  from which these boxes are removed has been given, we only need to specify the representation of  $S_n$ . This is given by specifying the number of rows with two boxes in the row  $= b_0$  and the number of rows with just a single box in the row  $= b_1$ . We have reserved  $O_c(b_0, 0)$  for  $O_d(b_0, b_1 = 0)$  because when  $b_1 = 0$  there is no corresponding  $O_e(b_0, b_1 = 0)$ .

Acting with the dilatation operator on these two column restricted Schur polynomials, one generates terms corresponding to Schur polynomials that have an  $R$  with three columns. The extra operators are shown below

[illegible]

In terms of these eight operators, the exact action of the dilatation operator is given in Appendix C.

The appearance of restricted Schur polynomials with three columns is a potential disaster. Indeed, the dilatation operator acting on these three column restricted Schur polynomials will produce four column restricted Schur polynomials; acting on these will produce five column restricted Schur polynomials and so on. This would imply that we would have to consider the dilatation operator acting on restricted Schur polynomials labeled by all possible Young diagrams  $R$  with  $n + 2$  boxes - a much more complicated problem. On physical grounds we would expect that at large  $N$  the threebrane number would be conserved - threebranes are stable semiclassical objects. Consequently, only two columns are long. Any extra columns are short. A restricted Schur polynomial with two long columns and some additional short columns corresponds to a bound state of two threebranes plus some KK gravitons. The transition from a two three-brane state to a state of two threebranes plus KK gravitons involves graviton emission so this transition amplitude will be proportional to the string coupling. In the 't Hooft limit,  $g_s \propto \frac{1}{N}$ , so this transition will be suppressed. Consequently, we did not expect mixing with three (or more) column restricted Schur polynomials. Inspecting the result of Appendix C, it's clear that the coefficients multiplying the three column contributions are of the same size as the coefficients multiplying the two column contributions. However, before we are able to decide whether terms are sub leading or not, we should



write all of our expressions in terms of operators normalized to have a unit two point function. The relevant two point functions were computed using the results of [13, 16] and are summarized in Appendix B. Notice that the three column terms are smaller by a factor of  $b_0$  as compared to the two column terms. This implies that the three column terms are suppressed by a factor of  $1/\sqrt{b_0}$  and may therefore be dropped if we take  $b_0 = O(N)$ . This is very natural. Indeed, in the case of the BMN loops we restrict the  $\mathcal{R}$ -charge  $J$  of the loop to be  $O(\sqrt{N})$  with  $J^2/N \ll 1$  so that mixing with multi-trace states is suppressed. Here we restrict the  $\mathcal{R}$ -charge of the two threebrane state to be large enough that  $b_0$  is  $O(N)$  so that mixing with more than two column states is suppressed.

We use hatted operators to denote normalized operators. The leading large  $N$  action of the dilatation operator on normalized operators is

$$\begin{aligned}
D\hat{O}_a(b_0, b_1) &= 4g_{YM}^2 \frac{(N - b_0 - b_1 - 1)}{(b_1 + 2)^2} \hat{O}_a(b_0, b_1) \\
&\quad - 2g_{YM}^2 \frac{\sqrt{(N - b_0 - b_1 - 1)(N - b_0 + 1)}}{b_1 + 2} \sqrt{\frac{b_1 + 3}{b_1 + 1}} \hat{O}_d(b_0, b_1) \\
&\quad + 2g_{YM}^2 \frac{\sqrt{(N - b_0 - b_1 - 1)(N - b_0 + 1)} b_1}{(b_1 + 2)^2} \sqrt{\frac{b_1 + 3}{b_1 + 1}} \hat{O}_e(b_0, b_1) \\
&\quad + 4g_{YM}^2 \frac{\sqrt{(N - b_0 - b_1 - 1)(N - b_0 + 1)}}{(b_1 + 2)^2} \hat{O}_b(b_0 - 1, b_1 + 2) \\
&\quad + 2g_{YM}^2 \frac{(N - b_0 - b_1 - 1)}{(b_1 + 2)} \sqrt{\frac{b_1 + 1}{b_1 + 3}} \hat{O}_d(b_0 - 1, b_1 + 2) \\
&\quad - 2g_{YM}^2 \frac{(b_1 + 4)(N - b_0 - b_1 - 1)}{(b_1 + 2)^2} \sqrt{\frac{b_1 + 1}{b_1 + 3}} \hat{O}_e(b_0 - 1, b_1 + 2) \\
\\
D\hat{O}_b(b_0, b_1) &= 4g_{YM}^2 \frac{\sqrt{(N - b_0)(N - b_0 - b_1)}}{b_1^2} \hat{O}_a(b_0 + 1, b_1 - 2) \\
&\quad - 2g_{YM}^2 \frac{(N - b_0)}{b_1} \sqrt{\frac{b_1 + 1}{b_1 - 1}} \hat{O}_d(b_0 + 1, b_1 - 2) \\
&\quad + 2g_{YM}^2 \frac{(N - b_0)(b_1 - 2)}{b_1^2} \sqrt{\frac{b_1 + 1}{b_1 - 1}} \hat{O}_e(b_0 + 1, b_1 - 2) \\
&\quad + 4g_{YM}^2 \frac{(N - b_0)}{b_1^2} \hat{O}_b(b_0, b_1)
\end{aligned}$$

$$\begin{aligned}
& +2g_{YM}^2 \frac{\sqrt{(N-b_0)(N-b_0-b_1)}}{b_1} \sqrt{\frac{b_1-1}{b_1+1}} \hat{O}_d(b_0, b_1) \\
& -2g_{YM}^2 \frac{(b_1+2)\sqrt{(N-b_0)(N-b_0-b_1)}}{b_1^2} \sqrt{\frac{b_1-1}{b_1+1}} \hat{O}_e(b_0, b_1)
\end{aligned}$$

$$\begin{aligned}
D\hat{O}_d(b_0, b_1) = & -2g_{YM}^2 \frac{\sqrt{(N-b_0+1)(N-b_0-b_1-1)}}{b_1+2} \sqrt{\frac{b_1+3}{b_1+1}} \hat{O}_a(b_0, b_1) \\
& +2g_{YM}^2 \frac{\sqrt{(N-b_0-b_1)(N-b_0)}}{b_1} \sqrt{\frac{b_1-1}{b_1+1}} \hat{O}_b(b_0, b_1) \\
& +g_{YM}^2 (2N-2b_0-b_1+3) \hat{O}_d(b_0, b_1) \\
& +g_{YM}^2 \frac{(N-b_0)(4-4b_1-2b_1^2)+b_1^3+b_1^2-4b_1}{b_1(b_1+2)} \hat{O}_e(b_0, b_1) \\
& -2g_{YM}^2 \frac{(N-b_0+1)}{b_1+2} \sqrt{\frac{b_1+3}{b_1+1}} \hat{O}_b(b_0-1, b_1+2) \\
& -g_{YM}^2 \sqrt{(N-b_0+1)(N-b_0-b_1-1)} \hat{O}_d(b_0-1, b_1+2) \\
& +g_{YM}^2 \frac{\sqrt{(N-b_0+1)(N-b_0-b_1-1)}(b_1+4)}{(b_1+2)} \hat{O}_e(b_0-1, b_1+2) \\
& +2g_{YM}^2 \frac{(N-b_0-b_1)}{b_1} \sqrt{\frac{b_1-1}{b_1+1}} \hat{O}_a(b_0+1, b_1-2) \\
& -g_{YM}^2 \sqrt{(N-b_0)(N-b_0-b_1)} \hat{O}_d(b_0+1, b_1-2) \\
& +g_{YM}^2 \frac{(b_1-2)\sqrt{(N-b_0)(N-b_0-b_1)}}{b_1} \hat{O}_e(b_0+1, b_1-2)
\end{aligned}$$

$$\begin{aligned}
D\hat{O}_e(b_0, b_1) = & 2g_{YM}^2 \frac{b_1\sqrt{(N-b_0+1)(N-b_0-b_1-1)}}{(b_1+2)^2} \sqrt{\frac{b_1+3}{b_1+1}} \hat{O}_a(b_0, b_1) \\
& -2g_{YM}^2 \frac{(b_1+2)\sqrt{(N-b_0-b_1)(N-b_0)}}{b_1^2} \sqrt{\frac{b_1-1}{b_1+1}} \hat{O}_b(b_0, b_1) \\
& +g_{YM}^2 \frac{(N-b_0)(4-2b_1^2-4b_1)+b_1^3+b_1^2-4b_1}{b_1(b_1+2)} \hat{O}_d(b_0, b_1) \\
& +g_{YM}^2 \frac{2(N-b_0)(b_1^4+4b_1^3+4b_1^2-8)-b_1^5-5b_1^4-8b_1^3+16b_1}{b_1^2(b_1+2)^2} \hat{O}_e(b_0, b_1)
\end{aligned}$$

$$\begin{aligned}
& +2g_{YM}^2 \frac{(N-b_0+1)b_1}{(b_1+2)^2} \sqrt{\frac{b_1+3}{b_1+1}} \hat{O}_b(b_0-1, b_1+2) \\
& +g_{YM}^2 \frac{\sqrt{(N-b_0+1)(N-b_0-b_1-1)}b_1}{(b_1+2)} \hat{O}_d(b_0-1, b_1+2) \\
& -g_{YM}^2 \frac{\sqrt{(N-b_0+1)(N-b_0-b_1-1)}b_1(b_1+4)}{(b_1+2)^2} \hat{O}_e(b_0-1, b_1+2) \\
& -2g_{YM}^2 \frac{(N-b_0-b_1)(b_1+2)}{b_1^2} \sqrt{\frac{b_1-1}{b_1+1}} \hat{O}_a(b_0+1, b_1-2) \\
& +g_{YM}^2 \frac{\sqrt{(N-b_0)(N-b_0-b_1)}(b_1+2)}{b_1} \hat{O}_d(b_0+1, b_1-2) \\
& -g_{YM}^2 \frac{(b_1+2)(b_1-2)\sqrt{(N-b_0)(N-b_0-b_1)}}{b_1^2} \hat{O}_e(b_0+1, b_1-2)
\end{aligned}$$

Notice that mixing is highly suppressed. Indeed, these operators are orthogonal, at tree level, to all order in  $1/N$ . From the above expression we see that at one loop operators can only mix if they differ by at most, by one box in their  $R$  label. This suppression has been observed before in other bases [14, 15, 27]. Another point worth noting is that if we take the usual 't Hooft limit  $N \rightarrow \infty$  with  $\lambda = g_{YM}^2 N$  fixed, the one loop anomalous dimension is  $O(1)$ . The usual 't Hooft limit thus leads to a well defined and non-trivial problem.

## 5. Emergence of the Radial Direction

Recall that the  $\mathcal{R}$ -charge of an operator in the field theory maps into the angular momentum of the dual string theory state and that the angular momentum of the string theory state determines its size. Identifying the two columns with the two threebranes, the number of boxes in each column determines the angular momentum and hence the size of each threebrane. In the limit that  $N - b_0 = O(N)$ ,  $b_0 = O(N)$  and  $b_1 = O(\sqrt{N})$  we have non-maximal giants which are separated by a distance of  $O(1)$  in string units. In this limit, we expect the dynamics to simplify. The action of the dilatation operator becomes

$$\begin{aligned}
D\hat{O}_a(b_0, b_1) &= \lambda \times O\left(\frac{1}{b_1}\right) \\
D\hat{O}_b(b_0, b_1) &= \lambda \times O\left(\frac{1}{b_1}\right)
\end{aligned}$$

$$D\hat{O}_d(b_0, b_1) = \lambda(1 - \frac{b_0}{N}) \left( 2\hat{O}_d(b_0, b_1) - \hat{O}_d(b_0 - 1, b_1 + 2) - \hat{O}_d(b_0 + 1, b_1 - 2) \right) \\ - \lambda(1 - \frac{b_0}{N}) \left( 2\hat{O}_e(b_0, b_1) - \hat{O}_e(b_0 - 1, b_1 + 2) - \hat{O}_e(b_0 + 1, b_1 - 2) \right) + \lambda \times O\left(\frac{1}{b_1}\right)$$

$$D\hat{O}_e(b_0, b_1) = \lambda(1 - \frac{b_0}{N}) \left( 2\hat{O}_e(b_0, b_1) - \hat{O}_e(b_0 - 1, b_1 + 2) - \hat{O}_e(b_0 + 1, b_1 - 2) \right) \\ - \lambda(1 - \frac{b_0}{N}) \left( 2\hat{O}_d(b_0, b_1) - \hat{O}_d(b_0 - 1, b_1 + 2) - \hat{O}_d(b_0 + 1, b_1 - 2) \right) + \lambda \times O\left(\frac{1}{b_1}\right)$$

These results have a natural interpretation. It looks as if  $\hat{O}_a(b_0, b_1)$ ,  $\hat{O}_b(b_0, b_1)$  and  $\hat{O}_d(b_0, b_1) + \hat{O}_e(b_0, b_1)$  remain supersymmetric. First, note that it is natural to interpret  $\hat{O}_a(b_0, b_1)$  as a state in which we deform only the smaller threebrane. Recall that deforming a single threebrane gives us a supersymmetric state so it seems natural for  $\hat{O}_a(b_0, b_1)$  to remain supersymmetric. Similarly,  $\hat{O}_b(b_0, b_1)$  can be interpreted as a state in which we deform only the smaller threebrane and a similar comment can be made. The fact that the combination  $\hat{O}_d(b_0, b_1) + \hat{O}_e(b_0, b_1)$  is annihilated by  $D$  suggests that there is also a supersymmetric way to deform the pair of threebranes. Finally, notice that if we set  $\hat{O}_d(b_0, b_1) - \hat{O}_e(b_0, b_1) \equiv \hat{O}_{d-e}(b_0, b_1)$  we have

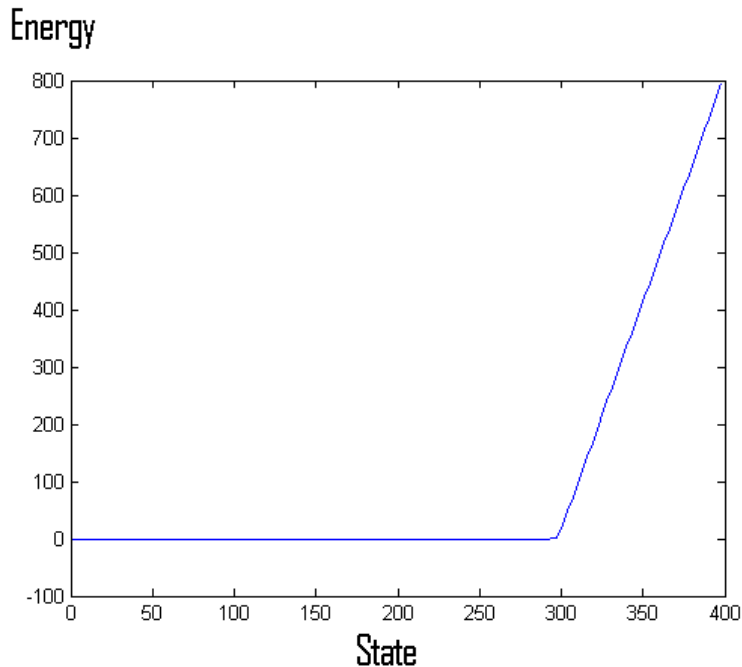
$$D\hat{O}_{d-e}(b_0, b_1) = 2\lambda(1 - \frac{b_0}{N}) \left( 2\hat{O}_{d-e}(b_0, b_1) - \hat{O}_{d-e}(b_0 - 1, b_1 + 2) - \hat{O}_{d-e}(b_0 + 1, b_1 - 2) \right).$$

The right hand side again looks like a discretization of the second derivative. This time *it is the Young diagram itself that is defining the lattice!* This result looks rather intuitive, especially after recalling that the number of boxes in each column sets the angular momentum and hence the radius of the corresponding threebrane.

## 6. Numerical Results

We have not managed to analytically solve for the spectrum of the one loop anomalous dimension. This would entail finding the eigenvalues and eigenvectors of the last equations given in section 4. It is however straight forward to solve for the spectrum numerically. The spectrum is given in figure 1.

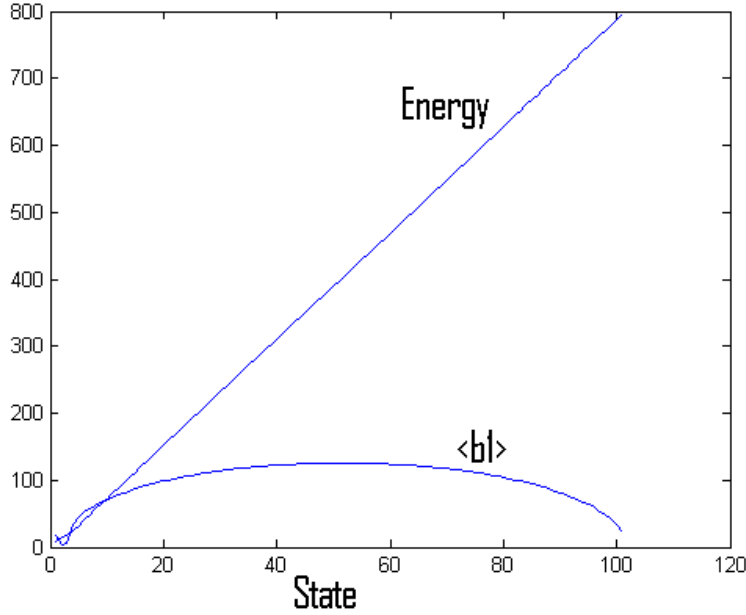
The example shown has  $N = 100000$  and considers operators built from 199800  $Z$ s and 2  $Y$ s. There are a total of 398 excited two threebrane states and 297 of them have



**Figure 1:** The spectrum of anomalous dimensions = the energy of the dual giant graviton state. To produce this plot we used  $N = 100000$  and considered an operator built from 199800  $Z$ s and 2  $Y$ s. There are a total of 398 excited two threebrane states.

zero energy. If the total number of states is  $N_{\text{states}}$  we find that  $0.75(N_{\text{states}} - 2)$  are massless. The results of the last section would have suggested that  $3/4$  of the states are zero energy, so this is not unexpected. What we did find rather remarkable, is that the states with non-zero energy have a constant energy spacing. This strongly suggests that this system is secretly a harmonic oscillator; we have not however been able to demonstrate this analytically. In the previous section we have seen that the equation for the non-zero energy states is written in terms of a discretized version of the second derivative. To obtain the spectrum we need to know what boundary conditions are to be applied. Notice that these boundary conditions are coded into the equations for the action of  $D$ . As  $b_1$  gets smaller and smaller the factors of  $b_1$  and  $b_1 - 1$  prevent the first column from shrinking to a size smaller than the second column. When the two columns are the same size, the first column can't shrink any further - this is one boundary. As the first column grows we get to a point where  $b_0 = N$ . At this point, the factors of  $N - b_0$  prevent the first column from growing to an even larger size. This is the second boundary.

The distance between the two threebranes is determined by the value of  $b_1$ . In figure



**Figure 2:** The spectrum of non-zero anomalous dimensions together with the expected value of  $b_1$ . To produce this plot we used  $N = 100000$  and considered an operator built from 199800  $Z$ s and 2  $Y$ s. There are a total of 398 excited two threebrane states.

2 we have plotted the non-zero energy eigenvalues together with the corresponding value of  $\langle b_1 \rangle$ . Notice that for a given value of  $\langle b_1 \rangle$  there are two energy eigenvalues. To interpret this note that each threebrane will behave like a harmonic oscillator; the open strings stretched between these threebranes implies that the oscillators are coupled. It is well known that coupled oscillators have two possible normal modes, corresponding to the oscillators oscillating in phase or out of phase. The lowest energy state corresponds to the oscillators oscillating in phase; in this mode the strings stretching between the threebranes will hardly be excited. When the oscillators oscillate out of phase, the strings stretching between the threebranes will be excited making this the mode with the largest energy. When the threebranes are very close to each other, they will join with “dimples” - they are too close for an actual open string to form. In this case, there is no open string to excite which corresponds to the tiny “V” on the lower left hand side of the plot. When an open string has formed, the energy difference between the two states will be determined by how strongly the open string and the threebrane couple. This coupling [22, 14, 15] has the form

$$\sqrt{1 - \frac{J}{N}}$$

where  $J$  is the angular momentum of the giant. As the larger giant becomes nearly maximal, this coupling switches off and the energy difference between the two states at fixed  $\langle b_1 \rangle$  decreases. The maximal giant is decoupled (at one loop and at large  $N$ ) from the open string. This is why, in figure 2 as  $\langle b_1 \rangle$  increases the difference in the energy of the two states decreases.

## 7. Conclusions

In this article we have determined the anomalous dimension of a class of operators whose classical dimension is  $O(N)$ . These operators have a dual interpretation as an excited two (threebrane) state. An analogous computation for BMN loops allows one to see the worldsheet emerge as a lattice defined by the matrices appearing in the loop. Our motivation was to see if an analogous result is possible for threebranes. Our answer is clearly “yes” with the lattice emerging from the Young diagram labeling the restricted Schur polynomial. This lattice is a discretization of the radial coordinate of the threebrane - different lattice points correspond to threebranes with different radii. Each threebrane behaves like a harmonic oscillator. The open strings stretched between these threebranes implies that the oscillators are coupled. Coupled oscillators have two possible normal modes, corresponding to the oscillators oscillating in phase or out of phase. These two modes are evident from the numerically computed spectrum of anomalous dimensions. Further, the known strength of the coupling between the open string and the threebrane is clearly evident in the numerical spectrum.

By taking the angular momenta (that is  $b_0$ ) of the threebranes to be at least  $O(N)$ , we have seen that in the large  $N$  limit there is a dynamical decoupling so that the restricted Schur’s with two columns do not mix with restricted Schur’s with a different number of columns. The number of columns can be identified with the number of threebranes, so that the dual statement is simply that at weak string coupling the threebrane number is conserved. The fact that this decoupling is achieved for certain values of the  $\mathcal{R}$ -charge is very similar to the BMN loop dynamics. The  $\mathcal{R}$ -charge of BMN loops  $J$  is chosen to satisfy  $J^2/N \ll 1$  to suppress mixing between single and multi traces. We have also seen that the usual ’t Hooft limit gives a well defined and non-trivial problem for the spectrum of the dilatation operator.

There are a number of natural extensions of the present study. We would like to generalize the computation here to an arbitrary number of  $Y$  fields, and then even further by using many types of impurities. The computation of the present paper has allowed a description of the radius of a giant graviton. It is only by considering the other complex Higgs scalars that we expect to see the worldvolume dimensions emerge. To consider general fluctuations of the giants it will be necessary to include both the

fermions and the gauge fields. Another natural question is if the dynamics for these excited threebranes is integrable or not.

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## A. Changing Basis, Character Identities

### A.1 Changing the basis

In these formulas  $\{\cdot\}$  stand for labels that specify the Young diagram: the number of boxes in the right most column (called  $b_0$ ) and the number of boxes in the left most column minus the number of boxes in the right most column (called  $b_1$ ). The weights<sup>3</sup>  $c_1, c_2$  are the weights of boxes  $a$  and  $b$  identified uniquely by the requirement  $c_1 > c_2$ . This fixes the weights to be

$$c_1 = N - b_0 + 1, \quad c_2 = N - b_0 - b_1,$$

when **the boxes are not in the same column**, or

$$c_1 = N - b_0 + 1, \quad c_2 = N - b_0,$$

when **the boxes are both in the right most column**, or

$$c_1 = N - b_0 - b_1, \quad c_2 = N - b_0 - b_1 - 1,$$

when **the boxes are both in the left most column**.

$$\begin{aligned} |a; \{\cdot\}\rangle &= \left| \begin{array}{c} \square \\ \square \\ \square \\ \square \end{array} ; \begin{array}{c} \square \\ \square \end{array} i, \begin{array}{c} \square \\ \square \end{array} \right\rangle = \left| \begin{array}{c} \square \\ \square \\ \square \\ \square \end{array} i, \begin{array}{c} \square \\ \square \end{array} \right\rangle = |1; \{\cdot\}\rangle, \\ |b; \{\cdot\}\rangle &= \left| \begin{array}{c} \square \\ \square \\ \square \\ \square \end{array} ; \begin{array}{c} \square \\ \square \end{array} i, \begin{array}{c} \square \\ \square \end{array} \right\rangle = \left| \begin{array}{c} \square \\ \square \\ \square \\ \square \end{array} i, \begin{array}{c} \square \\ \square \end{array} \right\rangle = |2; \{\cdot\}\rangle, \end{aligned}$$

---

<sup>3</sup>Recall that the weight of a box in row  $i$  and column  $j$  is  $N - i + j$ .



$$\begin{aligned}
|c; \{\cdot\}\rangle &= \left| \begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \square \\ \hline \square & \square \\ \hline \end{array}; \begin{array}{|c|} \hline \square \\ \hline \square \\ \hline \square \\ \hline \end{array} i, \square \right\rangle = \left| \begin{array}{|c|} \hline \square \\ \hline \square \\ \hline \square \\ \hline \end{array} i \right\rangle = |3; \{\cdot\}\rangle, \\
|d; \{\cdot\}\rangle &= \left| \begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \square \\ \hline \square & \square \\ \hline \end{array}; \begin{array}{|c|} \hline \square \\ \hline \square \\ \hline \square \\ \hline \end{array} i, \square \right\rangle = \sqrt{\frac{c_1 - c_2 + 1}{2(c_1 - c_2)}} \left| \begin{array}{|c|} \hline \square \\ \hline \square \\ \hline \square \\ \hline \end{array} i \right\rangle + \sqrt{\frac{c_1 - c_2 - 1}{2(c_1 - c_2)}} \left| \begin{array}{|c|} \hline \square \\ \hline \square \\ \hline \square \\ \hline \end{array} i \right\rangle \\
&= \sqrt{\frac{c_1 - c_2 + 1}{2(c_1 - c_2)}} |4; \{\cdot\}\rangle + \sqrt{\frac{c_1 - c_2 - 1}{2(c_1 - c_2)}} |5; \{\cdot\}\rangle, \\
|e; \{\cdot\}\rangle &= \left| \begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \square \\ \hline \square & \square \\ \hline \end{array}; \begin{array}{|c|} \hline \square \\ \hline \square \\ \hline \square \\ \hline \end{array} i, \square \right\rangle = \sqrt{\frac{c_1 - c_2 - 1}{2(c_1 - c_2)}} \left| \begin{array}{|c|} \hline \square \\ \hline \square \\ \hline \square \\ \hline \end{array} i \right\rangle - \sqrt{\frac{c_1 - c_2 + 1}{2(c_1 - c_2)}} \left| \begin{array}{|c|} \hline \square \\ \hline \square \\ \hline \square \\ \hline \end{array} i \right\rangle \\
&= \sqrt{\frac{c_1 - c_2 - 1}{2(c_1 - c_2)}} |4; \{\cdot\}\rangle - \sqrt{\frac{c_1 - c_2 + 1}{2(c_1 - c_2)}} |5; \{\cdot\}\rangle.
\end{aligned}$$

## A.2 Character Identities

The formulas of the previous subsection imply the following relations between characters

$$\chi_{\begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \square \\ \hline \square & \square \\ \hline \end{array}; \begin{array}{|c|} \hline \square \\ \hline \square \\ \hline \square \\ \hline \end{array}}(\sigma) = \chi_{\begin{array}{|c|} \hline \square \\ \hline \square \\ \hline \square \\ \hline \end{array}}(\sigma) \quad (\text{A.1})$$

$$\chi_{\begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \square \\ \hline \square & \square \\ \hline \end{array}; \begin{array}{|c|} \hline \square \\ \hline \square \\ \hline \square \\ \hline \end{array}}(\sigma) = \chi_{\begin{array}{|c|} \hline \square \\ \hline \square \\ \hline \square \\ \hline \end{array}}(\sigma) \quad (\text{A.2})$$

$$\chi_{\begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \square \\ \hline \square & \square \\ \hline \end{array}; \begin{array}{|c|} \hline \square \\ \hline \square \\ \hline \square \\ \hline \end{array}}(\sigma) = \chi_{\begin{array}{|c|} \hline \square \\ \hline \square \\ \hline \square \\ \hline \end{array}}(\sigma) \quad (\text{A.3})$$

$$\begin{aligned}
\chi_{\begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \square \\ \hline \square & \square \\ \hline \end{array}; \begin{array}{|c|} \hline \square \\ \hline \square \\ \hline \square \\ \hline \end{array}}(\sigma) &= \frac{b_1 + 2}{2(b_1 + 1)} \chi_{\begin{array}{|c|} \hline \square \\ \hline \square \\ \hline \square \\ \hline \end{array}}(\sigma) + \frac{b_1}{2(b_1 + 1)} \chi_{\begin{array}{|c|} \hline \square \\ \hline \square \\ \hline \square \\ \hline \end{array}}(\sigma) \\
&+ \frac{\sqrt{b_1(b_1 + 2)}}{2(b_1 + 1)} \chi_{\begin{array}{|c|} \hline \square \\ \hline \square \\ \hline \square \\ \hline \end{array}}(\sigma) + \frac{\sqrt{b_1(b_1 + 2)}}{2(b_1 + 1)} \chi_{\begin{array}{|c|} \hline \square \\ \hline \square \\ \hline \square \\ \hline \end{array}}(\sigma)
\end{aligned} \quad (\text{A.4})$$

$$\begin{aligned}
\chi_{\begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \square \\ \hline \square & \square \\ \hline \end{array}; \begin{array}{|c|} \hline \square \\ \hline \square \\ \hline \square \\ \hline \end{array}}(\sigma) &= \frac{b_1}{2(b_1 + 1)} \chi_{\begin{array}{|c|} \hline \square \\ \hline \square \\ \hline \square \\ \hline \end{array}}(\sigma) + \frac{b_1 + 2}{2(b_1 + 1)} \chi_{\begin{array}{|c|} \hline \square \\ \hline \square \\ \hline \square \\ \hline \end{array}}(\sigma) \\
&- \frac{\sqrt{b_1(b_1 + 2)}}{2(b_1 + 1)} \chi_{\begin{array}{|c|} \hline \square \\ \hline \square \\ \hline \square \\ \hline \end{array}}(\sigma) - \frac{\sqrt{b_1(b_1 + 2)}}{2(b_1 + 1)} \chi_{\begin{array}{|c|} \hline \square \\ \hline \square \\ \hline \square \\ \hline \end{array}}(\sigma)
\end{aligned} \quad (\text{A.5})$$

Finally, we will make use of the identities

$$\begin{aligned}
\chi_{\begin{array}{|c|} \hline \square \\ \square \\ \square \\ \hline \end{array}}((n, n+2)\sigma - \sigma(n, n+2)) &= \frac{1}{b_1+3} \sqrt{1 - \frac{1}{(b_1+2)^2}} \left[ \chi_{\begin{array}{|c|} \hline \square & \square \\ \square & \square \\ \square & \square \\ \hline \end{array}}(\sigma) - \chi_{\begin{array}{|c|} \hline \square & \square \\ \square & \square \\ \square & \square \\ \hline \end{array}}(\sigma) \right] \\
&- \sqrt{1 - \frac{1}{(b_1+2)^2}} \sqrt{1 - \frac{1}{(b_1+3)^2}} \left[ \chi_{\begin{array}{|c|} \hline \square & \square \\ \square & \square \\ \square & \square \\ \hline \end{array}}(\sigma) - \chi_{\begin{array}{|c|} \hline \square & \square \\ \square & \square \\ \square & \square \\ \hline \end{array}}(\sigma) \right] \\
\chi_{\begin{array}{|c|} \hline \square & \square \\ \square & \square \\ \square & \square \\ \hline \end{array}}((n, n+2)\sigma - \sigma(n, n+2)) &= -\frac{1}{b_1-1} \sqrt{1 - \frac{1}{(b_1)^2}} \left[ \chi_{\begin{array}{|c|} \hline \square & \square \\ \square & \square \\ \square & \square \\ \hline \end{array}}(\sigma) - \chi_{\begin{array}{|c|} \hline \square & \square \\ \square & \square \\ \square & \square \\ \hline \end{array}}(\sigma) \right] \\
&- \sqrt{1 - \frac{1}{(b_1-1)^2}} \sqrt{1 - \frac{1}{(b_1)^2}} \left[ \chi_{\begin{array}{|c|} \hline \square & \square \\ \square & \square \\ \square & \square \\ \hline \end{array}}(\sigma) - \chi_{\begin{array}{|c|} \hline \square & \square \\ \square & \square \\ \square & \square \\ \hline \end{array}}(\sigma) \right] \\
\chi_{\begin{array}{|c|} \hline \square & \square \\ \square & \square \\ \square & \square \\ \hline \end{array}}((n, n+2)\sigma - \sigma(n, n+2)) &= -\frac{\sqrt{3}}{2} \left[ \chi_{\begin{array}{|c|} \hline \square & \square \\ \square & \square \\ \square & \square \\ \hline \end{array}}(\sigma) - \chi_{\begin{array}{|c|} \hline \square & \square \\ \square & \square \\ \square & \square \\ \hline \end{array}}(\sigma) \right] \\
\chi_{\begin{array}{|c|} \hline \square & \square \\ \square & \square \\ \square & \square \\ \hline \end{array}}((n, n+2)\sigma - \sigma(n, n+2)) &= \frac{1}{b_1+1} \sqrt{1 - \frac{1}{(b_1)^2}} \left[ \chi_{\begin{array}{|c|} \hline \square & \square \\ \square & \square \\ \square & \square \\ \hline \end{array}}(\sigma) - \chi_{\begin{array}{|c|} \hline \square & \square \\ \square & \square \\ \square & \square \\ \hline \end{array}}(\sigma) \right] \\
&- \frac{1}{b_1} \sqrt{1 - \frac{1}{(b_1+1)^2}} \left[ \chi_{\begin{array}{|c|} \hline \square & \square \\ \square & \square \\ \square & \square \\ \hline \end{array}}(\sigma) - \chi_{\begin{array}{|c|} \hline \square & \square \\ \square & \square \\ \square & \square \\ \hline \end{array}}(\sigma) \right]
\end{aligned}$$

$$\begin{aligned}
& + \frac{1}{b_1 + 2} \sqrt{1 - \frac{1}{(b_1 + 1)^2}} \left[ \chi_{\begin{array}{|c|} \hline \square \\ \square \\ \square \\ \square \\ \square \\ \hline \end{array}}(\sigma) - \chi_{\begin{array}{|c|} \hline \square \\ \square \\ \square \\ \square \\ \square \\ \square \\ \hline \end{array}}(\sigma) \right] \\
& - \sqrt{1 - \frac{1}{(b_1 + 1)^2}} \sqrt{1 - \frac{1}{(b_1 + 2)^2}} \left[ \chi_{\begin{array}{|c|} \hline \square \\ \square \\ \square \\ \square \\ \square \\ \square \\ \square \\ \hline \end{array}}(\sigma) - \chi_{\begin{array}{|c|} \hline \square \\ \square \\ \square \\ \square \\ \square \\ \square \\ \square \\ \square \\ \hline \end{array}}(\sigma) \right] \\
& \chi_{\begin{array}{|c|} \hline \square \\ \square \\ \square \\ \square \\ \square \\ \square \\ \square \\ \square \\ \hline \end{array}}((n, n + 2)\sigma - \sigma(n, n + 2)) = -\frac{1}{b_1} \sqrt{1 - \frac{1}{(b_1 + 1)^2}} \left[ \chi_{\begin{array}{|c|} \hline \square \\ \square \\ \square \\ \square \\ \square \\ \square \\ \square \\ \square \\ \square \\ \hline \end{array}}(\sigma) - \chi_{\begin{array}{|c|} \hline \square \\ \square \\ \square \\ \square \\ \square \\ \square \\ \square \\ \square \\ \square \\ \square \\ \hline \end{array}}(\sigma) \right] \\
& - \sqrt{1 - \frac{1}{(b_1)^2}} \sqrt{1 - \frac{1}{(b_1 + 1)^2}} \left[ \chi_{\begin{array}{|c|} \hline \square \\ \square \\ \square \\ \square \\ \square \\ \square \\ \square \\ \square \\ \square \\ \square \\ \square \\ \hline \end{array}}(\sigma) - \chi_{\begin{array}{|c|} \hline \square \\ \square \\ \square \\ \square \\ \square \\ \square \\ \square \\ \square \\ \square \\ \square \\ \square \\ \square \\ \hline \end{array}}(\sigma) \right] \\
& + \frac{1}{b_1 + 1} \sqrt{1 - \frac{1}{(b_1 + 2)^2}} \left[ \chi_{\begin{array}{|c|} \hline \square \\ \square \\ \square \\ \square \\ \square \\ \square \\ \square \\ \square \\ \square \\ \square \\ \square \\ \square \\ \square \\ \hline \end{array}}(\sigma) - \chi_{\begin{array}{|c|} \hline \square \\ \square \\ \square \\ \square \\ \square \\ \square \\ \square \\ \square \\ \square \\ \square \\ \square \\ \square \\ \square \\ \square \\ \hline \end{array}}(\sigma) \right] \\
& + \frac{1}{b_1 + 2} \sqrt{1 - \frac{1}{(b_1 + 1)^2}} \left[ \chi_{\begin{array}{|c|} \hline \square \\ \square \\ \square \\ \square \\ \square \\ \square \\ \square \\ \square \\ \square \\ \square \\ \square \\ \square \\ \square \\ \square \\ \square \\ \hline \end{array}}(\sigma) - \chi_{\begin{array}{|c|} \hline \square \\ \square \\ \square \\ \square \\ \square \\ \square \\ \square \\ \square \\ \square \\ \square \\ \square \\ \square \\ \square \\ \square \\ \square \\ \square \\ \hline \end{array}}(\sigma) \right]
\end{aligned}$$

### A.3 New Identities

In this subsection we will derive identities that will allow us to express terms involving a restricted Schur polynomial in  $Y^2$  or a trace of  $Y$  times a restricted Schur polynomial, as a linear combination of restricted Schur polynomials in  $Y$ . Start from the expression

$$\chi_{R', R''}(Z, Y^2) = \frac{1}{n!} \sum_{\sigma \in S_{n+1}} \chi_{R', R''}(\sigma) Z_{i_{\sigma(1)}}^{i_1} \cdots Z_{i_{\sigma(n)}}^{i_n} (Y^2)_{i_{\sigma(n+1)}}^{i_{n+1}}$$

$$\begin{aligned}
&= \frac{1}{n!} \sum_{\sigma \in S_{n+1}} \chi_{R', R''}(\sigma) Z_{i_{\sigma(1)}}^{i_1} \cdots Z_{i_{\sigma(n)}}^{i_n} Y_{i_{n+2}}^{i_{n+1}} Y_{i_{\sigma(n+1)}}^{i_{n+2}} \\
&= \frac{1}{n!} \sum_{\sigma \in S_{n+1}} \chi_{R', R''}(\sigma) \text{Tr}(\sigma(n+1, n+2) Z^{\otimes n} Y^{\otimes 2}) \\
&= \frac{1}{n!} \sum_{\sigma \in S_{n+1}} \chi_{R', R''}(\sigma) \sum_{R, (r_1, r_2)} \chi^{R, (r_1, r_2)}(\sigma(n+1, n+2)) \chi_{R, (r_1, r_2)}(Z, Y) \\
&= \sum_{R, (r_1, r_2)} \alpha_{R, (r_1, r_2)} \chi_{R, (r_1, r_2)}(Z, Y).
\end{aligned}$$

In the above,  $\chi^{R, (r_1, r_2)}(\sigma(n+1, n+2))$  is the dual character, computed and defined in [32]. It is given by

$$\chi^{R, (r_1, r_2)}(\sigma(n+1, n+2)) = \frac{d_R n! 2!}{d_{r_1} d_{r_2} (n+2)!} \chi_{R, (r_1, r_2)}(\sigma(n+1, n+2)).$$

The coefficients  $\alpha_{R, (r_1, r_2)}$  are

$$\begin{aligned}
\alpha_{R, (r_1, r_2)} &= \frac{1}{n!} \sum_{\sigma \in S_{n+1}} \chi_{R', R''}(\sigma) \chi^{R, (r_1, r_2)}(\sigma(n+1, n+2)) \\
&= \frac{d_R 2!}{d_{r_1} d_{r_2} (n+2)!} \sum_{\sigma \in S_{n+1}} \chi_{R', R''}(\sigma) \chi_{R, (r_1, r_2)}(\sigma(n+1, n+2)).
\end{aligned}$$

Lets do the sum over  $\sigma$  by rewriting this last expression as

$$\alpha_{R, (r_1, r_2)} = \frac{d_R 2!}{d_{r_1} d_{r_2} (n+2)!} \text{Tr}_{R, (r_1, r_2)} \left( \sum_{\sigma \in S_{n+1}} \chi_{R', R''}(\sigma) \sigma(n+1, n+2) \right).$$

The reason why this is a useful rewriting is that it is easy to recognize that

$$\sum_{\sigma \in S_{n+1}} \chi_{R', R''}(\sigma) \sigma = \frac{(n+1)!}{d_{R'}} P_{R', R''}$$

is  $P_{R', R''}$ , a projection operator acting on  $R'$ , projecting to  $R''$ . To see that the LHS is a projector, the reader is encouraged to verify that, for example, it squares to itself and that if one sums over  $R'$  and  $R''$ , one recovers the standard projector onto irreducible representation  $R$ . Finally, she can also check that it annihilates states not in the  $R''$  subspace. Thus, we obtain (use  $d_{r_2} = 1$ )

$$\alpha_{R, (r_1, r_2)} = \frac{2 \text{hooks}_{R'}}{\text{hooks}_R d_{r_1}} \sum_{i, j} \langle R, (r_1, r_2); i | R', R''; j \rangle \langle R', R''; j | (n+1, n+2) | R, (r_1, r_2); i \rangle$$

which is a lovely explicit expression that is easy to evaluate.

It is equally easy to argue that

$$\chi_{R', R''}(Z, Y) \text{Tr}(Y) = \sum_{R, (r_1, r_2)} \beta_{R, (r_1, r_2)} \chi_{R, (r_1, r_2)}(Z, Y),$$

where

$$\beta_{R, (r_1, r_2)} = \frac{2 \text{hooks}_{R'}}{\text{hooks}_R d_{r_1}} \sum_{i, j} \langle R, (r_1, r_2); i | R', R''; j \rangle \langle R', R''; j | R, (r_1, r_2); i \rangle.$$

Applying these formulas we obtain (the box with a star on the LHS contains the  $Y^2$ )

$$\begin{aligned} \chi_{\begin{array}{|c|} \hline \square \\ \square \\ \square \\ \hline \end{array}} = & \frac{1}{b_0 + b_1 + 2} \chi_{\begin{array}{|c|} \hline \square \\ \square \\ \square \\ \hline \end{array}} \begin{array}{|c|} \hline \square \\ \square \\ \hline \end{array} - \frac{b_1 + 2}{b_1(b_0 + b_1 + 2)} \chi_{\begin{array}{|c|} \hline \square \\ \square \\ \square \\ \hline \end{array}} \begin{array}{|c|} \hline \square \\ \square \\ \hline \end{array} - \frac{2(b_1 - 1)}{b_1(b_0 + 2)} \chi_{\begin{array}{|c|} \hline \square \\ \square \\ \square \\ \hline \end{array}} \begin{array}{|c|} \hline \square \\ \square \\ \hline \end{array} \\ & + \frac{b_0 + b_1 + 1}{b_0 + b_1 + 2} \chi_{\begin{array}{|c|} \hline \square \\ \square \\ \square \\ \hline \end{array}} \begin{array}{|c|} \hline \square \\ \square \\ \hline \end{array} - \frac{b_0(b_0 + b_1 + 1)}{(b_0 + 2)(b_0 + b_1 + 2)} \chi_{\begin{array}{|c|} \hline \square \\ \square \\ \square \\ \hline \end{array}} \begin{array}{|c|} \hline \square \\ \square \\ \hline \end{array} \end{aligned}$$

and (the box with the star on the LHS contains the  $Y$ )

$$\begin{aligned} \text{Tr}(Y) \chi_{\begin{array}{|c|} \hline \square \\ \square \\ \square \\ \hline \end{array}} = & \frac{1}{b_0 + b_1 + 2} \chi_{\begin{array}{|c|} \hline \square \\ \square \\ \square \\ \hline \end{array}} \begin{array}{|c|} \hline \square \\ \square \\ \hline \end{array} + \frac{b_1 + 2}{b_1(b_0 + b_1 + 2)} \chi_{\begin{array}{|c|} \hline \square \\ \square \\ \square \\ \hline \end{array}} \begin{array}{|c|} \hline \square \\ \square \\ \hline \end{array} + \frac{2(b_1 - 1)}{b_1(b_0 + 2)} \chi_{\begin{array}{|c|} \hline \square \\ \square \\ \square \\ \hline \end{array}} \begin{array}{|c|} \hline \square \\ \square \\ \hline \end{array} \\ & + \frac{b_0 + b_1 + 1}{b_0 + b_1 + 2} \chi_{\begin{array}{|c|} \hline \square \\ \square \\ \square \\ \hline \end{array}} \begin{array}{|c|} \hline \square \\ \square \\ \hline \end{array} + \frac{b_0(b_0 + b_1 + 1)}{(b_0 + 2)(b_0 + b_1 + 2)} \chi_{\begin{array}{|c|} \hline \square \\ \square \\ \square \\ \hline \end{array}} \begin{array}{|c|} \hline \square \\ \square \\ \hline \end{array} \end{aligned}$$

and (the box with the star on the LHS contains the  $Y^2$ )

$$\begin{aligned} \chi_{\begin{array}{|c|} \hline \square \\ \square \\ \square \\ \hline \end{array}} = & \frac{1}{b_0 + 1} \chi_{\begin{array}{|c|} \hline \square \\ \square \\ \square \\ \hline \end{array}} \begin{array}{|c|} \hline \square \\ \square \\ \hline \end{array} - \frac{b_1}{(b_0 + 1)(b_1 + 2)} \chi_{\begin{array}{|c|} \hline \square \\ \square \\ \square \\ \hline \end{array}} \begin{array}{|c|} \hline \square \\ \square \\ \hline \end{array} - \frac{2(b_1 + 3)}{(b_1 + 2)(b_0 + b_1 + 3)} \chi_{\begin{array}{|c|} \hline \square \\ \square \\ \square \\ \hline \end{array}} \begin{array}{|c|} \hline \square \\ \square \\ \hline \end{array} \\ & + \frac{b_0}{b_0 + 1} \chi_{\begin{array}{|c|} \hline \square \\ \square \\ \square \\ \hline \end{array}} \begin{array}{|c|} \hline \square \\ \square \\ \hline \end{array} - \frac{b_0}{b_0 + 1} \frac{b_0 + b_1 + 1}{b_0 + b_1 + 3} \chi_{\begin{array}{|c|} \hline \square \\ \square \\ \square \\ \hline \end{array}} \begin{array}{|c|} \hline \square \\ \square \\ \hline \end{array} \end{aligned}$$

and (the box with the star on the LHS contains the  $Y$ )

$$\begin{aligned}
\text{Tr}(Y)\chi_{\begin{array}{|c|} \hline \square \\ \hline \end{array}} &= \frac{1}{b_0+1}\chi_{\begin{array}{|c|c|} \hline \square & \square \\ \hline \end{array}} + \frac{b_1}{(b_0+1)(b_1+2)}\chi_{\begin{array}{|c|c|} \hline \square & \square \\ \hline \end{array}} + \frac{2(b_1+3)}{(b_1+2)(b_0+b_1+3)}\chi_{\begin{array}{|c|c|} \hline \square & \square \\ \hline \end{array}} \\
&+ \frac{b_0}{b_0+1}\chi_{\begin{array}{|c|c|} \hline \square & \square \\ \hline \end{array}} + \frac{b_0}{b_0+1}\frac{b_0+b_1+1}{b_0+b_1+3}\chi_{\begin{array}{|c|c|} \hline \square & \square \\ \hline \end{array}}.
\end{aligned}$$

These are all of the identities that we will need.

## B. Normalization Factors

Normalized operators are normalized to have a unit two point function. Thus, to compute the normalization factor of any operator we simply need to compute its two point function. In this appendix we will compute the two point functions of all operators that appear. We use the notation  $f_R$  to denote the product of all the weights in Young diagram  $R$ . Further, we list the row lengths to specify the Young diagram. Thus, for example,  $f_{(3,2^{b_0-1},1^{b_1+1})}$  is the product of weights in the Young diagram that has one row with 3 boxes in it,  $b_0 - 1$  rows with 2 boxes and  $b_1 + 1$  rows with a single box. The two point functions we need are

$$\langle O_a(b_0, b_1)O_a^\dagger(b_0, b_1) \rangle = f_{(2^{b_0}, 1^{b_1+2})} \frac{(b_1+1)(b_0+b_1+2)(b_0+b_1+3)}{2(b_1+3)}$$

$$\langle O_b(b_0, b_1)O_b^\dagger(b_0, b_1) \rangle = f_{(2^{b_0+2}, 1^{b_1-2})} \frac{(b_1+1)(b_0+1)(b_0+2)}{2(b_1-1)}$$

$$\langle O_c(b_0, b_1)O_c^\dagger(b_0, b_1) \rangle = f_{(2^{b_0+1})} \frac{(b_0+1)(b_0+2)}{2}$$

$$\langle O_d(b_0, b_1)O_d^\dagger(b_0, b_1) \rangle = f_{(2^{b_0+1}, 1^{b_1})} \frac{(b_0+1)(b_0+b_1+2)}{2} = \langle O_e(b_0, b_1)O_e^\dagger(b_0, b_1) \rangle$$

$$\langle O_f(b_0, b_1)O_f^\dagger(b_0, b_1) \rangle = f_{(3, 2^{b_0-1}, 1^{b_1+1})} \frac{(b_0+1)(b_0+b_1+3)(b_1+1)}{2b_0(b_1+2)} = \langle O_g(b_0, b_1)O_g^\dagger(b_0, b_1) \rangle$$

$$\left\langle O_h(b_0, b_1) O_h^\dagger(b_0, b_1) \right\rangle = f_{(3, 2^{b_0}, 1^{b_1-1})} \frac{(b_0+2)(b_0+b_1+2)(b_1+1)}{2b_1(b_0+b_1+1)} = \left\langle O_i(b_0, b_1) O_i^\dagger(b_0, b_1) \right\rangle$$

In the limit we consider  $O_f$ ,  $O_g$ ,  $O_h$  and  $O_i$  are all subleading. This is crucial to show that we do indeed have a subsector that decouples dynamically.

### C. Exact Action of the One Loop Dilatation Operator

The action of the dilatation operator given in this section was checked by evaluating both sides numerically with randomly generated  $Z$  and  $Y$ , for the case that  $R$  contains 5 or 6 boxes.

$$\begin{aligned} DO_a(b_0, b_1) = & 4 \frac{(N - b_0 - b_1 - 1)}{(b_1 + 2)^2} \left( 1 - \frac{b_1 + 3}{b_0 + b_1 + 3} \right) O_a(b_0, b_1) \\ & - 2 \frac{(N - b_0 - b_1 - 1)}{b_1 + 2} \left( 1 - \frac{1}{b_0 + 1} \right) O_d(b_0, b_1) \\ & + 2 \frac{(N - b_0 - b_1 - 1)b_1}{(b_1 + 2)^2} \left( 1 - \frac{1}{b_0 + 1} \right) O_e(b_0, b_1) \\ & + 4 \frac{(b_1 + 1)(N - b_0 - b_1 - 1)}{(b_1 + 3)(b_1 + 2)^2} \left( 1 + \frac{(b_1 + 1)}{(b_0 + 1)} \right) O_b(b_0 - 1, b_1 + 2) \\ & + 2 \frac{(b_1 + 1)(N - b_0 - b_1 - 1)}{(b_1 + 2)(b_1 + 3)} \left( 1 - \frac{1}{b_0 + b_1 + 3} \right) O_d(b_0 - 1, b_1 + 2) \\ & - 2 \frac{(b_1 + 4)(b_1 + 1)(N - b_0 - b_1 - 1)}{(b_1 + 3)(b_1 + 2)^2} \left( 1 - \frac{1}{(b_0 + b_1 + 3)} \right) O_e(b_0 - 1, b_1 + 2) \\ & + 2 \frac{(N - b_0 - b_1 - 1)}{b_1 + 2} \frac{b_0}{b_0 + 1} O_f(b_0, b_1) - 2 \frac{(N - b_0 - b_1 - 1)}{b_1 + 2} \frac{b_0}{b_0 + 1} \frac{b_0 + b_1 + 1}{b_0 + b_1 + 3} O_g(b_0, b_1) \\ & - 2 \frac{(N - b_0 - b_1 - 1)(b_1 + 1)}{(b_1 + 2)(b_1 + 3)} \frac{b_0 + b_1 + 2}{b_0 + b_1 + 3} O_h(b_0 - 1, b_1 + 2) \\ & + 2 \frac{(N - b_0 - b_1 - 1)(b_1 + 1)}{(b_1 + 2)(b_1 + 3)} \frac{(b_0 - 1)(b_0 + b_1 + 2)}{(b_0 + 1)(b_0 + b_1 + 3)} O_i(b_0 - 1, b_1 + 2). \end{aligned}$$

$$DO_b(b_0, b_1) = 4 \frac{(N - b_0)(b_1 + 1)}{(b_1 - 1)b_1^2} \left( 1 - \frac{b_1 + 1}{b_0 + b_1 + 2} \right) O_a(b_0 + 1, b_1 - 2)$$

$$\begin{aligned}
& -2 \frac{(N-b_0)(b_1+1)}{b_1(b_1-1)} \left(1 - \frac{1}{b_0+2}\right) O_d(b_0+1, b_1-2) \\
& +2 \frac{(N-b_0)(b_1+1)(b_1-2)}{b_1^2(b_1-1)} \left(1 - \frac{1}{b_0+2}\right) O_e(b_0+1, b_1-2) \\
& +4 \frac{(N-b_0)}{b_1^2} \left(1 + \frac{b_1-1}{b_0+2}\right) O_b(b_0, b_1) \\
& +2 \frac{N-b_0}{b_1} \left(1 - \frac{1}{b_0+b_1+2}\right) O_d(b_0, b_1) \\
& -2 \frac{(b_1+2)(N-b_0)}{b_1^2} \left(1 - \frac{1}{b_0+b_1+2}\right) O_e(b_0, b_1) \\
& +2 \frac{(N-b_0)(b_1+1)}{b_1(b_1-1)} \frac{b_0+1}{b_0+2} O_f(b_0+1, b_1-2) -2 \frac{(N-b_0)(b_1+1)}{b_1(b_1-1)} \frac{b_0+1}{b_0+2} \frac{b_0+b_1}{b_0+b_1+2} O_g(b_0+1, b_1-2) \\
& -2 \frac{N-b_0}{b_1} \frac{b_0+b_1+1}{b_0+b_1+2} O_h(b_0, b_1) +2 \frac{N-b_0}{b_1} \frac{b_0(b_0+b_1+1)}{(b_0+2)(b_0+b_1+2)} O_i(b_0, b_1) .
\end{aligned}$$

$$\begin{aligned}
DO_d(b_0, b_1) &= -2 \frac{(b_1+3)(N-b_0+1)}{(b_1+1)(b_1+2)} \left(1 - \frac{b_1+3}{b_0+b_1+3}\right) O_a(b_0, b_1) \\
& +2 \frac{(b_1-1)(N-b_0-b_1)}{b_1(b_1+1)} \left(1 + \frac{b_1-1}{b_0+2}\right) O_b(b_0, b_1) \\
& + \left(2N - 2b_0 - b_1 + 3 - \frac{(N-b_0+1)(b_1+3)}{(b_0+1)(b_1+1)} - \frac{(b_1-1)(N-b_0-b_1)}{(b_1+1)(b_0+b_1+2)}\right) O_d(b_0, b_1) \\
& + \left(\frac{(N-b_0)(4-4b_1-2b_1^2)+b_1^3+b_1^2-4b_1}{b_1(b_1+2)} + \frac{b_1(b_1+3)(N-b_0+1)}{(b_0+1)(b_1+1)(b_1+2)}\right. \\
& \quad \left.+ \frac{(b_1-1)(N-b_0-b_1)(b_1+2)}{(b_1+1)b_1(b_0+b_1+2)}\right) O_e(b_0, b_1) \\
& -2 \frac{(N-b_0+1)}{b_1+2} \left(1 + \frac{b_1+1}{b_0+1}\right) O_b(b_0-1, b_1+2) \\
& -(N-b_0+1) \left(1 - \frac{1}{b_0+b_1+3}\right) O_d(b_0-1, b_1+2) \\
& + \frac{(N-b_0+1)(b_1+4)}{(b_1+2)} \left(1 - \frac{1}{b_0+b_1+3}\right) O_e(b_0-1, b_1+2) \\
& +2 \frac{(N-b_0-b_1)}{b_1} \left(1 - \frac{b_1+1}{b_0+b_1+2}\right) O_a(b_0+1, b_1-2)
\end{aligned}$$



$$\begin{aligned}
& -(N - b_0 - b_1) \left( 1 - \frac{1}{b_0 + 2} \right) O_d(b_0 + 1, b_1 - 2) \\
& + \frac{(b_1 - 2)(N - b_0 - b_1)}{b_1} \left( 1 - \frac{1}{b_0 + 2} \right) O_e(b_0 + 1, b_1 - 2) \\
& - \frac{(b_1 + 3)(N - b_0 + 1)}{b_1 + 1} \frac{b_0}{b_0 + 1} O_f(b_0, b_1) + \frac{(b_1 + 3)(N - b_0 + 1)}{b_1 + 1} \frac{b_0}{b_0 + 1} \frac{b_0 + b_1 + 1}{b_0 + b_1 + 3} O_g(b_0, b_1) \\
& + (N - b_0 + 1) \frac{b_0 + b_1 + 2}{b_0 + b_1 + 3} O_h(b_0 - 1, b_1 + 2) - (N - b_0 + 1) \frac{(b_0 - 1)(b_0 + b_1 + 2)}{(b_0 + 1)(b_0 + b_1 + 3)} O_i(b_0 - 1, b_1 + 2) \\
& - \frac{(b_1 - 1)(N - b_0 - b_1)}{(b_1 + 1)} \frac{b_0 + b_1 + 1}{b_0 + b_1 + 2} O_h(b_0, b_1) + \frac{(b_1 - 1)(N - b_0 - b_1)}{b_1 + 1} \frac{b_0(b_0 + b_1 + 1)}{(b_0 + 2)(b_0 + b_1 + 2)} O_i(b_0, b_1) \\
& + (N - b_0 - b_1) \frac{b_0 + 1}{b_0 + 2} O_f(b_0 + 1, b_1 - 2) - (N - b_0 - b_1) \frac{b_0 + 1}{b_0 + 2} \frac{b_0 + b_1}{b_0 + b_1 + 2} O_g(b_0 + 1, b_1 - 2)
\end{aligned}$$

$$\begin{aligned}
DO_e(b_0, b_1) &= 2 \frac{(b_1 + 3)b_1(N - b_0 + 1)}{(b_1 + 1)(b_1 + 2)^2} \left( 1 - \frac{b_1 + 3}{b_0 + b_1 + 3} \right) O_a(b_0, b_1) \\
& - 2 \frac{(b_1 - 1)(b_1 + 2)(N - b_0 - b_1)}{b_1^2(b_1 + 1)} \left( 1 + \frac{b_1 - 1}{b_0 + 2} \right) O_b(b_0, b_1) \\
& + \left( \frac{(N - b_0)(4 - 2b_1^2 - 4b_1) + b_1^3 + b_1^2 - 4b_1}{b_1(b_1 + 2)} + \frac{(b_1 + 3)(N - b_0 + 1)b_1}{(b_0 + 1)(b_1 + 1)(b_1 + 2)} \right. \\
& \quad \left. + \frac{(b_1 - 1)(b_1 + 2)(N - b_0 - b_1)}{b_1(b_1 + 1)(b_0 + b_1 + 2)} \right) O_d(b_0, b_1) \\
& + \left( \frac{2(N - b_0)(b_1^4 + 4b_1^3 + 4b_1^2 - 8) - b_1^5 - 5b_1^4 - 8b_1^3 + 16b_1}{b_1^2(b_1 + 2)^2} - \frac{b_1^2(b_1 + 3)(N - b_0 + 1)}{(b_0 + 1)(b_1 + 1)(b_1 + 2)^2} \right. \\
& \quad \left. - \frac{(b_1 - 1)(b_1 + 2)(N - b_0 - b_1)(b_1 + 2)}{b_1^2(b_1 + 1)(b_0 + b_1 + 2)} \right) O_e(b_0, b_1) \\
& + 2 \frac{(N - b_0 + 1)b_1}{(b_1 + 2)^2} \left( 1 + \frac{b_1 + 1}{b_0 + 1} \right) O_b(b_0 - 1, b_1 + 2) \\
& + \frac{(N - b_0 + 1)b_1}{(b_1 + 2)} \left( 1 - \frac{1}{b_0 + b_1 + 3} \right) O_d(b_0 - 1, b_1 + 2) \\
& - \frac{(N - b_0 + 1)b_1(b_1 + 4)}{(b_1 + 2)^2} \left( 1 - \frac{1}{b_0 + b_1 + 3} \right) O_e(b_0 - 1, b_1 + 2) \\
& - 2 \frac{(N - b_0 - b_1)(b_1 + 2)}{b_1^2} \left( 1 - \frac{b_1 + 1}{b_0 + b_1 + 2} \right) O_a(b_0 + 1, b_1 - 2)
\end{aligned}$$

$$\begin{aligned}
& + \frac{(N - b_0 - b_1)(b_1 + 2)}{b_1} \left(1 - \frac{1}{b_0 + 2}\right) O_d(b_0 + 1, b_1 - 2) \\
& - \frac{(b_1 + 2)(b_1 - 2)(N - b_0 - b_1)}{b_1^2} \left(1 - \frac{1}{b_0 + 2}\right) O_e(b_0 + 1, b_1 - 2) \\
& + \frac{(b_1 + 3)(N - b_0 + 1)b_1}{(b_1 + 1)(b_1 + 2)} \frac{b_0}{b_0 + 1} O_f(b_0, b_1) - \frac{(b_1 + 3)(N - b_0 + 1)b_1}{(b_1 + 1)(b_1 + 2)} \frac{b_0(b_0 + b_1 + 1)}{(b_0 + 1)(b_0 + b_1 + 3)} O_g(b_0, b_1) \\
& - \frac{b_1(N - b_0 + 1)}{b_1 + 2} \frac{b_0 + b_1 + 2}{b_0 + b_1 + 3} O_h(b_0 - 1, b_1 + 2) \\
& + \frac{b_1(N - b_0 + 1)}{b_1 + 2} \frac{(b_0 + b_1 + 2)(b_0 - 1)}{(b_0 + b_1 + 3)(b_0 + 1)} O_i(b_0 - 1, b_1 + 2) \\
& + \frac{(b_1 - 1)(b_1 + 2)(N - b_0 - b_1)}{b_1(b_1 + 1)} \frac{b_0 + b_1 + 1}{b_0 + b_1 + 2} O_h(b_0, b_1) \\
& - \frac{(b_1 - 1)(b_1 + 2)(N - b_0 - b_1)b_0(b_0 + b_1 + 1)}{b_1(b_1 + 1)(b_0 + 2)(b_0 + b_1 + 2)} O_i(b_0, b_1) \\
& - \frac{(b_1 + 2)(N - b_0 - b_1)}{b_1} \frac{b_0 + 1}{b_0 + 2} O_f(b_0 + 1, b_1 - 2) \\
& + \frac{(b_1 + 2)(N - b_0 - b_1)}{b_1} \frac{b_0 + 1}{b_0 + 2} \frac{b_0 + b_1}{b_0 + b_1 + 2} O_g(b_0 + 1, b_1 - 2)
\end{aligned}$$

These formulas are exact. They will, of course, simplify dramatically once we drop subleading terms.

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